

Spiky Strings with Two Spins in AdS_5

Shijong Ryang

*Department of Physics
Kyoto Prefectural University of Medicine
Taishogun, Kyoto 603-8334 Japan
ryang@koto.kpu-m.ac.jp*

Abstract

Using the reduction of the string sigma model to the 1-d Neumann integrable model we reconstruct the closed string solution with two equal spins in AdS_5 which is specified by the number of round arcs and one winding number. From the string sigma model itself as well as its reduction to the Neumann-Rosochatius system we construct a spiky string solution with two unequal spins in AdS_5 whose ratio is fixed and derive its energy-spin relation. The string configuration is characterized by the number of spikes and two equal winding numbers associated with the two rotating angular directions.

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1 Introduction

The AdS/CFT correspondence [1] has more and more revealed the deep relations between the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and the string theory in $AdS_5 \times S^5$, where various types of classical string solutions play an important role. The energy spectrum of certain string states matches with the spectrum of dimensions of field theory operators in the SYM theory. There has been a mounting evidence that the spectrum of AdS/CFT is described by studying the multi-spin folded or circular rotating string solutions in $AdS_5 \times S^5$ in a particular large spin limit [2, 3, 4] and by analyzing the Bethe equation for the diagonalization of the integrable spin chain in the planar SYM theory [5, 6].

Specially the energy of the one-spin folded rotating string in AdS_3 [2, 7] describes the dimension of twist two gauge theory operator. The large spin limit of it is related [8] via an analytic continuation and an $SO(2,4)$ transformation to the open string solution ending on a null cusp at the boundary which determines the planar four-point gluon amplitude at strong coupling in the SYM theory [9, 10, 11, 12, 13]. The quantum corrections to the folded string energy have been computed up to two loops in the long string limit [3, 8, 14, 15, 16] and have been tested to match with the prediction of the strong-coupling expansion of the solution [17] of the integral equation for the minimal twist anomalous dimensions as extracted from the weak-coupling all-loop asymptotic Bethe ansatz [18, 6]. This matching has been extended to include the spin J in $S^1 \subset S^5$ [19, 20].

The spiky string solution in AdS_3 has been constructed by using the Nambu-Goto string action as a generalization of the one-spin folded string solution to the case of many spikes [21]. This string state has been argued to correspond to a subclass of higher twist operators in the $SL(2)$ sector of gauge theory [22]. The angular separation between each of adjacent spikes is the same. Using the string sigma-model action in the conformal gauge the Kruczenski's spiky string solution with one spin has been reproduced in [23]. The spiky string solution in AdS_3 with different angular separations has been constructed [24] by using the finite-gap formalism for the string sigma-model action [22, 25], where each classical solution of string theory has an associated spectral elliptic curve which encodes the values of the higher conserved charges of the worldsheet sigma model. A different attempt to generalize the spiky string solution has been presented [26], where a construction of a general class of string solution in AdS_3 is based on a Pohlmeyer type reduction [27] with the sinh-Gordon model, whose N-soliton solutions map to the dynamical N-spike AdS string configurations.

For the spiky string in AdS_3 in the large spin limit the spikes approach the boundary of AdS_3 and their motion can be described by a string solution in an AdS_3 -pp-wave metric [28]. Further various solutions for open strings moving in the AdS_3 -pp-wave space have been produced such that they are dual to various Wilson loops in the field theory in the boundary 4-dimensional pp-wave background [29]. There has been an investigation of the spiky string solution in $AdS_3 \times S^1$ with n spikes and spin S in AdS_3 and spin J and winding m in S^1 [30]. In a special large n limit the solution can be regarded as describing a periodic-spike string in AdS_3 -pp-wave $\times S^1$ background. On the other hand based on the $SL(2)$ asymptotic Bethe ansatz equations, the rational (S, J) solution has been constructed as the one-cut solution with non-trivial winding [31] and the spiky string solution with winding m , spikes n and two spins S, J has been described as the two-cut solution [32] by following the general procedure

of solving integral Bethe ansatz equation at strong coupling [33]. The energy of this two-cut solution matches with that of the spiky string solution presented in [30] in a special scaling limit with large two spins and large winding where the string touches the AdS_5 boundary.

The single-spin folded string in AdS_3 has been extended so that the string solutions with two equal spins $S_1 = S_2$ in AdS_5 [34] and further with three spins $S_1 = S_2$ and J in $AdS_5 \times S^1$ [35] are produced by using string sigma-model actions in the conformal gauge, where the closed string configurations consisting of round arcs are characterized by the number of arcs n and the winding number m . The large spin behavior of the minimal energy specified by $n = 3$, $m = 1$ for the string solution with two equal spins $S_1 = S_2$ in AdS_5 has been argued [36] to match with the strong-coupling prediction from the analysis [37] of the full asymptotic Bethe ansatz equation [6, 18]. In the approach of [38] that the conformal-gauge string sigma model is reduced to the 1-d Neumann integrable model, the closed string solutions in AdS_5 are parametrized by the three frequencies $\omega_a = (\omega_0, \omega_1, \omega_2)$, $\omega_0 = \kappa$ and the two integrals of motion b_1, b_2 . In the special $\kappa = \omega_2 \neq \omega_1$ case an explicit solution with two unequal spins $S_1 \neq S_2$ has been constructed by making suitable change of global coordinates for the conformal-gauge string sigma-model action itself [36]. Further for the other special $b_1 = b_2$ case the energy-spin relation of circular string solution with two unequal spins has been derived by analyzing the 1-d Neumann integrable model.

Using the approach of [39] that the conformal-gauge string sigma model is reduced to the integrable Neumann-Rosochatius system, a solution which expresses a hanging string moving with two spins in AdS_5 has been presented [40] and shown to correspond to the double-helix Wilson loop describing a quark and an anti-quark moving on an S^3 .

We will consider the 1-d Neumann integrable model for the $\omega_1 = \omega_2$ case of the closed string in AdS_5 to reconstruct the string solution of round arc shape with two equal spins $S_1 = S_2$. In order to extend the spiky string in AdS_3 to in AdS_5 we will derive a spiky string solution with two spins by analyzing the conformal-gauge string sigma model itself as well as its reduction to the integrable Neumann-Rosochatius system. The energy-spin relations for both string solutions in the large spin limit will be compared.

2 Strings of round arc shape with two equal spins

We consider a rigid rotating two-spin string in AdS_5 based on the reduction to the 1-d Neumann integrable model [38]. For the closed bosonic string in the conformal gauge in AdS_5 space parametrized by global coordinates

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2) \quad (1)$$

we make the following ansatz

$$t = \kappa\tau, \quad \rho = \rho(\sigma), \quad \theta = \theta(\sigma), \quad \phi_1 = \omega_1\tau, \quad \phi_2 = \omega_2\tau. \quad (2)$$

In terms of the three complex embedding coordinates X_a , $a = 0, 1, 2$ subject to a constraint

$$-1 = |X_1|^2 + |X_2|^2 - |X_3|^2 = \sum_a \eta_a X_a \bar{X}_a \quad (3)$$

with $\eta_0 = -1, \eta_1 = \eta_2 = 1$, the AdS_5 metric (1) is rewritten by

$$ds^2 = \sum_a \eta_a dX_a d\bar{X}_a, \quad (4)$$

where

$$\begin{aligned} X_0 &= r_0 e^{it}, & X_1 &= r_1 e^{i\phi_1}, & X_2 &= r_2 e^{i\phi_2}, \\ r_0 &= \cosh \rho, & r_1 &= \sinh \rho \sin \theta, & r_2 &= \sinh \rho \cos \theta. \end{aligned} \quad (5)$$

Following the approach of ref. [38, 36] we use the two independent coordinates ζ_i , $i = 1, 2$ related to r_a as

$$r_a^2 = -\eta_a \frac{(\zeta_1 - \omega_a^2)(\zeta_2 - \omega_a^2)}{\prod_{b \neq a} \omega_{ab}^2}, \quad (6)$$

where $\omega_a = (\omega_0, \omega_1, \omega_2)$, $\omega_0 = \kappa$ and $\omega_{ab}^2 \equiv \omega_a^2 - \omega_b^2$. In terms of the coordinates $\zeta_i(\sigma)$ the string equations of motion read

$$\zeta_1'^2 = -4 \frac{P_5(\zeta_1)}{(\zeta_2 - \zeta_1)^2}, \quad \zeta_2'^2 = -4 \frac{P_5(\zeta_2)}{(\zeta_2 - \zeta_1)^2}, \quad (7)$$

where a quintic polynomial is expressed as

$$P_5(\zeta) = (\zeta - \omega_0^2)(\zeta - \omega_1^2)(\zeta - \omega_2^2)(\zeta - b_1)(\zeta - b_2) \quad (8)$$

with two constants of motion b_1, b_2 obeying

$$b_1 + b_2 = \omega_0^2 + \omega_1^2 + \omega_2^2. \quad (9)$$

The b_1, b_2 parameters are associated with the integrals of motion of the 1-d Neumann integrable model. The two-spin solution of a circular shape is specified by the following range

$$\omega_0^2 \leq \omega_2^2 \leq \zeta_1 \leq \omega_1^2 \leq b_1 \leq \zeta_2 \leq b_2. \quad (10)$$

We consider the $\omega_1 = \omega_2$ case corresponding to the rotating string with two equal spins in AdS_5 . For the equations in (7) we make the change of variables

$$\zeta_1 = \omega_1^2 - \omega_{12}^2 \xi_1, \quad \zeta_2 = b_2 - b_{21} \xi_2 \quad (11)$$

with $b_{21} \equiv b_2 - b_1$ and then take the limit $\omega_2 \rightarrow \omega_1$ to obtain the equations of motion for ξ_1, ξ_2

$$\xi_1'^2 = 4h(1-u) \left(1 - \frac{t}{u}\right) \frac{\xi_1(1-\xi_1)}{(1-u\xi_2)^2}, \quad (12)$$

$$\xi_2'^2 = 4h\xi_2(1-\xi_2)(1-t\xi_2), \quad (13)$$

where the parameters h, u and t are given by

$$h = b_2 - \omega_0^2 > 0, \quad u = \frac{b_{21}}{b_2 - \omega_1^2} > 0, \quad t = \frac{b_{21}}{b_2 - \omega_0^2} > 0. \quad (14)$$

The string coordinates r_a in the limit $\omega_2 \rightarrow \omega_1$ are expressed in terms of ξ_1, ξ_2 as

$$\begin{aligned} r_0^2 &= \frac{b_2 - \omega_0^2}{\omega_{10}^2} (1 - t\xi_2), \\ r_1^2 &= \frac{b_2 - \omega_1^2}{\omega_{10}^2} \xi_1 (1 - u\xi_2), \quad r_2^2 = \frac{b_2 - \omega_1^2}{\omega_{10}^2} (1 - \xi_1)(1 - u\xi_2). \end{aligned} \quad (15)$$

The equation (13) is integrated to be expressed in terms of the Jacobi elliptic function sn and the complete elliptic integral of the first kind K

$$\xi_2(\sigma) = \text{sn}^2(K(t) - \sqrt{h}\sigma, t), \quad (16)$$

where we choose that $\xi_2(\sigma)$ decreases in the interval $0 < \sigma < \pi/2$ and fix an integration constant as $\xi_2(0) = 1$ so that $\xi_2(\sigma_0) = 0$ with $\sigma_0 \equiv K(t)/\sqrt{h}$. The substitution of the solution (16) into r_0^2 in (15) yields

$$\cosh^2 \rho = \cosh^2 \rho_1 \text{dn}^2(K - \sqrt{h}\sigma, t) \quad (17)$$

with $\cosh^2 \rho_1 = (b_2 - \omega_0^2)/\omega_{10}^2$ so that $\cosh^2 \rho(\sigma_0) = \cosh^2 \rho_1$. We define $\cosh^2 \rho_0 = (b_1 - \omega_0^2)/\omega_{10}^2$ to have $\cosh^2 \rho(0) = \cosh^2 \rho_0$. The interval $b_1 \leq \zeta_2 \leq b_2$ in (10) implies $\rho_0 \leq \rho_1$. We see that $\rho(\sigma)$ starts off at its minimum ρ_0 at $\sigma = 0$, then increases to its maximum ρ_1 at $\sigma = \sigma_0$ in one segment. By gluing together $2n_1$ such segments that are n_1 arcs to impose the periodicity condition $\rho(\sigma + 2\pi) = \rho(\sigma)$, we have

$$\sqrt{h} \frac{2\pi}{n_1} = 2K(t) \quad (18)$$

because the period of dn is $2K$. The expression (17) is rewritten by

$$\cosh^2 \rho = \frac{\cosh^2 \rho_0}{\text{dn}^2(\sqrt{h}\sigma, t)}, \quad (19)$$

where t and \sqrt{h} are described in terms of $\cosh \rho_1, \cosh \rho_0$ as

$$t = \frac{\cosh^2 \rho_1 - \cosh^2 \rho_0}{\cosh^2 \rho_1}, \quad \sqrt{h} = \sqrt{\omega_{10}^2} \cosh \rho_1. \quad (20)$$

The other equation (12) is also integrated to be

$$\xi_1 = \sin^2 \left(\sqrt{h(1-u)} \left(1 - \frac{t}{u} \right) \int_{\sigma_0}^{\sigma} \frac{d\sigma}{1 - u\xi_2} \right). \quad (21)$$

Since the string configuration (15) implies

$$\tan^2 \theta = \frac{r_1^2}{r_2^2} = \frac{\xi_1}{1 - \xi_1} \quad (22)$$

we have

$$\theta(\sigma) = \sqrt{h(1-u) \left(1 - \frac{t}{u}\right)} \int_{\sigma_0}^{\sigma} \frac{d\sigma}{1 - u\xi_2}, \quad (23)$$

where an integration constant is chosen such that $\theta(\sigma = \sigma_0) = 0$ (i.e. $\theta(\rho = \rho_1) = 0$) for convenience. Substituting the solution (16) into (23) and integrating we derive

$$\theta(\sigma) = -\sqrt{\frac{\cosh^2 \rho_0 - 1}{\cosh^2 \rho_1 (\cosh^2 \rho_1 - 1)}} \Pi(\text{am}(K - \sqrt{h}\sigma), u, t) \quad (24)$$

with $u = (\cosh^2 \rho_1 - \cosh^2 \rho_0)/(\cosh^2 \rho_1 - 1)$, where $\Pi(x, u, t)$ is the incomplete of elliptic integral of the third kind. From the periodicity condition $\theta(2\pi) - \theta(0) = 2\pi n_2$ for the angular coordinate an integer winding number n_2 is specified by

$$2\pi n_2 = 2n_1(\theta(\sigma_0) - \theta(0)) \equiv 2n_1 \Delta\theta, \quad n_2 = 1, 2, 3, \dots, \quad (25)$$

which together with (24) yields

$$\frac{n_2}{n_1} \pi = \sqrt{\frac{\cosh^2 \rho_0 - 1}{\cosh^2 \rho_1 (\cosh^2 \rho_1 - 1)}} \Pi(u, t) \equiv \Delta\theta. \quad (26)$$

For the $\omega_1 \neq \omega_2$ case we write down the energy and two spins of rotating string

$$\begin{aligned} \mathcal{E} &= \omega_0 \int_0^{2\pi} \frac{d\sigma}{2\pi} r_0^2 = \frac{\omega_0(b_2 - \omega_0^2)}{\omega_{20}^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} (1 - t\xi_2) \left(1 - \frac{\omega_{12}^2}{\omega_{10}^2} \xi_1\right), \\ \mathcal{S}_1 &= \omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} r_1^2 = \frac{\omega_1(b_2 - \omega_1^2)}{\omega_{10}^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \xi_1 \left(1 - \frac{b_{21}}{b_2 - \omega_1^2} \xi_2\right), \\ \mathcal{S}_2 &= \omega_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} r_2^2 = \frac{\omega_2(b_2 - \omega_2^2)}{\omega_{20}^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} (1 - \xi_1) \left(1 - \frac{b_{21}}{b_2 - \omega_2^2} \xi_2\right). \end{aligned} \quad (27)$$

For the $\omega_1 = \omega_2$ case the energy and total spin are simplified to be

$$\begin{aligned} \mathcal{E} &= \frac{\omega_0(b_2 - \omega_0^2)}{\omega_{10}^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} (1 - t\xi_2), \\ \mathcal{S} &= \mathcal{S}_1 + \mathcal{S}_2 = 2\mathcal{S}_1 = \frac{\omega_1(b_2 - \omega_1^2)}{\omega_{10}^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} (1 - u\xi_2). \end{aligned} \quad (28)$$

Here from (14) we obtain b_1, b_2 as

$$b_1 = \frac{t(1-u)\omega_0^2 + u(t-1)\omega_1^2}{t-u}, \quad b_2 = \frac{t\omega_0^2 - u\omega_1^2}{t-u}, \quad (29)$$

which combine with (9) to give

$$\frac{\omega_0^2}{\omega_1^2} = \frac{t(2-u)}{u+t-ut}. \quad (30)$$

Gathering (18), (20) and (30) together we have

$$\omega_1 = \frac{\sqrt{\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1}}{\cosh \rho_1} \frac{n_1 K(t)}{\pi}. \quad (31)$$

Therefore the two constants of motion b_1, b_2 can be expressed in terms of ρ_0, ρ_1 . We use (30) and the explicit expression ξ_2 of (16) to derive

$$\begin{aligned} \mathcal{E} &= \frac{n_1}{\pi} \frac{\omega_0}{\omega_{10}^2} \sqrt{h} E(t) = \frac{n_1}{\pi} \sqrt{ut(2-u)} \frac{E(t)}{u-t}, \\ \mathcal{S} &= \frac{n_1}{\pi} \frac{\omega_1}{\sqrt{h}} \left(\frac{uE(t)}{u-t} - K(t) \right) = \frac{n_1}{\pi} \sqrt{\frac{u+t-ut}{u}} \left(\frac{uE(t)}{u-t} - K(t) \right), \end{aligned} \quad (32)$$

where E is the complete elliptic integral of the second kind.

Thus we have reproduced the same expressions for $\mathcal{E}, \mathcal{S}, \Delta\theta$ as in [34, 35, 36] by means of the reduction to the 1-d Neumann integrable model. This equal two-spin string configuration consists of n_1 round arcs with equal angular separation $2\Delta\theta$. In the large spin limit $u \approx t \approx 1$ in $t \leq u$ which is associated with $\omega_0 \leq \omega_1$ in (30), these expressions yield the following energy-spin relation

$$E - S \approx \frac{n_1 \sqrt{\lambda}}{2\pi} \left(\ln \frac{16\pi S}{n_1 \sqrt{\lambda}} - 1 + 2 \ln \sin \frac{n_2 \pi}{n_1} \right), \quad (33)$$

where $\sqrt{\lambda}$ is the string tension and the subleading contributions are presented in [36].

3 Spiky strings with two spins

We consider a spiky string with two spins in AdS_5 in the global coordinates of (1) using the string sigma-model action in the conformal gauge. We choose equal two frequencies $\omega_1 = \omega_2 = \omega$ and parametrize the closed string as

$$\begin{aligned} t &= \omega_0 \tau + \mu_0(\sigma), & \phi_1 &= \omega \tau + \mu_1(\sigma), & \phi_2 &= \omega \tau + \mu_2(\sigma), \\ \rho &= \rho(\sigma), & \theta &= \theta(\sigma). \end{aligned} \quad (34)$$

The equations of motion for t, ϕ_1 and ϕ_2 lead to

$$\mu'_0 = -\frac{C_0}{\cosh^2 \rho}, \quad \mu'_1 = \frac{C_1}{\sinh^2 \rho \sin^2 \theta}, \quad \mu'_2 = \frac{C_2}{\sinh^2 \rho \cos^2 \theta} \quad (35)$$

with three integration constants C_0, C_1, C_2 . The Virasoro constraints read

$$\omega_0 C_0 + \omega(C_1 + C_2) = 0, \quad (36)$$

$$\begin{aligned} \rho'^2 &+ \sinh^2 \rho \theta'^2 - \cosh^2 \rho \left(\omega_0^2 + \frac{C_0^2}{\cosh^4 \rho} \right) \\ &+ \sinh^2 \rho \left(\omega^2 + \frac{C_1^2}{\sinh^4 \rho \sin^2 \theta} + \frac{C_2^2}{\sinh^4 \rho \cos^2 \theta} \right) = 0. \end{aligned} \quad (37)$$

In view of the equation of motion for θ

$$\frac{\partial}{\partial \sigma}(\sinh^2 \rho \theta') - \frac{\sin \theta \cos \theta}{\sinh^2 \rho} \left(\frac{C_1^2}{\sin^4 \theta} - \frac{C_2^2}{\cos^4 \theta} \right) = 0 \quad (38)$$

we obtain a solution $\theta = \theta_0$ with

$$\tan^2 \theta_0 = \frac{C_1}{C_2}, \quad (39)$$

whose special case is $\theta_0 = \pi/4, C_1 = C_2$. Then the Virasoro constraint (37) is expressed in a factorized form as

$$\rho'^2 = \frac{(\omega_0^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho) \left(\sinh^2 2\rho - \frac{4C_0^2}{\omega^2} \right)}{\sinh^2 2\rho}, \quad (40)$$

where we use two relations $C_1 = -\omega_0 C_0 \sin^2 \theta_0 / \omega, C_2 = -\omega_0 C_0 \cos^2 \theta_0 / \omega$ derived from (36) and (39).

Alternatively we analyze this spiky string by using the reduction to the integrable Neumann-Rosochatius system [39, 40]. The closed string is parametrized in terms of the embedding coordinates X_a in (3), (4) as

$$X_a = r_a e^{i(\mu_a(\sigma) + \omega_a \tau)}, \quad a = 0, 1, 2 \quad (41)$$

with $\omega_1 = \omega_2 = \omega$ and $\sum_a \eta_a r_a^2 = -1$. Following the prescription for the string in AdS_5 presented in [40], the functions $\mu_a(\sigma)$ are characterized by

$$\mu'_a = \frac{\eta_a C_a}{r_a^2}. \quad (42)$$

One of the Virasoro constraints yields

$$\sum_a \omega_a C_a = 0, \quad (43)$$

which is the same as (36). This system is described by the two unconstrained coordinates $\zeta_i(\sigma)$, $i = 1, 2$ in (6) and solved by using the Hamiltonian-Jacobi method. The string equations of motion are also expressed by (7) but the quintic polynomial $P_5(\zeta)$ is given by

$$P_5(\zeta) = - \prod_a (\zeta - \omega_a^2) \left(V - \sum_a \prod_{b \neq a} (\omega_a^2 - \omega_b^2) \frac{C_a^2}{\zeta - \omega_a^2} + \zeta \sum_a \omega_a^2 - \zeta^2 \right), \quad (44)$$

which satisfies $P_5(\zeta_i) < 0$ and is compared with (8). The parameter V is a constant of motion and C_a are the conserved momenta canonically conjugate to μ_a in the integrable Neumann-Rosochatius system.

From $r_a^2 > 0$ in (6) the relevant parameters take the following range

$$\omega_0^2 \leq \omega_2^2 \leq \zeta_1 \leq \omega_1^2 \leq \zeta_2 \quad (45)$$

in the same way as (10). We take the limit $\omega_2 \rightarrow \omega_1$ after the change of variable $\zeta_1 = \omega_1^2 - \omega_{12}^2 \xi_1$. The equation of motion for ζ_1 is described in terms of ξ_1 as

$$\xi_1'^2 = -\frac{4\xi_1(1-\xi_1)\omega_{10}^2}{(\zeta_2 - \omega_1^2)^2} \left(V + \left(\frac{C_1^2}{\xi_1} + \frac{C_2^2}{1-\xi_1} - C_0^2 \right) \omega_{10}^2 + (\omega_0^2 + \omega_1^2)\omega_1^2 \right). \quad (46)$$

In order to find a solution with constant ξ_1 using (22) and (43) we express the relevant terms in (46) as

$$\frac{C_1^2}{\xi_1} + \frac{C_2^2}{1-\xi_1} = \left(\frac{\omega_0 C_0}{\omega_1} \right)^2 \frac{1 + \tan^2 \theta \left(\frac{C_2}{C_1} \right)^2}{\sin^2 \theta \left(1 + \frac{C_2}{C_1} \right)^2}, \quad (47)$$

which becomes $(\omega_0 C_0 / \omega_1)^2$ if we choose θ to be the same as (39). The string configuration with fixed θ_0 can be a solution when the following condition is satisfied

$$\omega_1^2 V + \omega_1^4 (\omega_0^2 + \omega_1^2) = C_0^2 (\omega_{10}^2)^2. \quad (48)$$

In the equation of motion for ζ_2 the $P_5(\zeta_2)$ is expressed owing to (48) as $P_5(\zeta_2) = (\zeta_2 - \omega_1^2)(\zeta_2 - \omega_2^2)f(\zeta_2)$ with

$$f(\zeta_2) = \zeta_2^3 - 2(\omega_0^2 + \omega_1^2)\zeta_2^2 + (\omega_0^2(\omega_0^2 + 2\omega_1^2) - V)\zeta_2 + (\omega_0^2 + \omega_1^2)(V + \omega_1^4), \quad (49)$$

which has three roots

$$\begin{aligned} \lambda_1 &= \frac{1}{2} \left(\omega_0^2 + \omega_1^2 - \sqrt{(\omega_0^2 + \omega_1^2)^2 + 4(V + \omega_1^4)} \right), \\ \lambda_2 &= \frac{1}{2} \left(\omega_0^2 + \omega_1^2 + \sqrt{(\omega_0^2 + \omega_1^2)^2 + 4(V + \omega_1^4)} \right), \\ \lambda_3 &= \omega_0^2 + \omega_1^2, \end{aligned} \quad (50)$$

so that $f(\zeta_2) = (\zeta_2 - \lambda_1)(\zeta_2 - \lambda_2)(\zeta_2 - \lambda_3)$. The λ_1, λ_2 are real since the square root term is given by $\omega_{10}^2 \sqrt{1 + 4(C_0/\omega_1)^2}$. The condition (48) implies $V + \omega_1^2(\omega_0^2 + \omega_1^2) > 0$ which further leads to $f(\omega_0^2) = \omega_1^2(V + \omega_0^2\omega_1^2 + \omega_1^4) > 0$, $f(\omega_1^2) = \omega_0^2(V + \omega_0^2\omega_1^2 + \omega_1^4) > 0$. We consider the case that the parameter V changes in

$$V \leq -\omega_1^4 \quad (51)$$

such that $\lambda_1 < \lambda_2 < \lambda_3$. Therefore taking account of $P_5(\zeta_2) < 0$, that is $f(\zeta_2) < 0$ and

$$\lambda_1 + \lambda_2 = \omega_0^2 + \omega_1^2 \quad (52)$$

we have the following parameter region

$$\lambda_1 \leq \omega_0^2 \leq \omega_1^2 \leq \lambda_2 \leq \zeta_2 \leq \lambda_3. \quad (53)$$

The parametrization of ζ_2 as $\zeta_2 = \lambda_3 - \lambda_{32}\xi_2$ with $\lambda_{32} \equiv \lambda_3 - \lambda_2$ makes the equation of motion for ζ_2 change into

$$\xi_2'^2 = 4h\xi_2(1-\xi_2)(1-t\xi_2) \quad (54)$$

with $h = \lambda_{31} \equiv \lambda_3 - \lambda_1 > 0$, $t = \lambda_{32}/\lambda_{31} > 0$. This equation also gives a solution

$$\xi_2(\sigma) = \text{sn}^2(K(t) - \sqrt{h}\sigma, t). \quad (55)$$

The string coordinates r_a (6) in the limit $\omega_2 \rightarrow \omega_1$ are expressed in terms of $\xi_1 = \sin^2 \theta_0$ and $\xi_2(\sigma)$ as

$$\begin{aligned} r_0^2 &= \frac{\lambda_3 - \omega_0^2}{\omega_{10}^2} (1 - u_0 \xi_2), \\ r_1^2 &= \frac{\lambda_3 - \omega_1^2}{\omega_{10}^2} \xi_1 (1 - u_1 \xi_2), \quad r_2^2 = \frac{\lambda_3 - \omega_1^2}{\omega_{10}^2} (1 - \xi_1) (1 - u_1 \xi_2) \end{aligned} \quad (56)$$

with $u_0 = \lambda_{32}/(\lambda_3 - \omega_0^2) > 0$, $u_1 = \lambda_{32}/(\lambda_3 - \omega_1^2) > 0$.

From (56) with (55) the AdS radial coordinate ρ is determined by

$$\begin{aligned} \cosh^2 \rho &= \frac{\lambda_3 - \omega_0^2 - \lambda_{32} \text{sn}^2(K - \sqrt{h}\sigma, t)}{\omega_{10}^2} \\ &= \frac{\cosh^2 \rho_1 \text{dn}^2(\sqrt{h}\sigma, t) - (\cosh^2 \rho_1 - \cosh^2 \rho_0) \text{cn}^2(\sqrt{h}\sigma, t)}{\text{dn}^2(\sqrt{h}\sigma, t)}, \end{aligned} \quad (57)$$

which is also represented by

$$\cosh 2\rho = \cosh 2\rho_1 \text{cn}^2(K - \sqrt{h}\sigma, t) + \cosh 2\rho_0 \text{sn}^2(K - \sqrt{h}\sigma, t), \quad (58)$$

where

$$\cosh^2 \rho_1 = \frac{\lambda_3 - \omega_0^2}{\omega_{10}^2} = \frac{\omega_1^2}{\omega_{10}^2}, \quad \cosh^2 \rho_0 = \frac{\lambda_2 - \omega_0^2}{\omega_{10}^2}. \quad (59)$$

In one segment the radial coordinate ρ starts off at its minimum ρ_0 at $\sigma = 0$ and increases to its maximum ρ_1 at $\sigma = \sigma_0 \equiv K(t)/\sqrt{h}$. The gluing of $2n_1$ segments to form a closed string yields a condition

$$\sqrt{h}\pi = n_1 K(t), \quad (60)$$

where t is expressed through a relation (52) as

$$t = \frac{\cosh^2 \rho_1 - \cosh^2 \rho_0}{\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1} = \frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 + \cosh 2\rho_0}, \quad (61)$$

which is compared with the expression of t in (20).

From the explicit solution (55) the equations (42) for $\mu_a(\sigma)$ are integrated to be

$$\begin{aligned} \mu_0 &= -\frac{C_0 \omega_{10}^2}{\lambda_3 - \omega_0^2} \int_{\sigma_0}^{\sigma} \frac{d\sigma}{1 - u_0 \xi_2} = \frac{C_0}{\cosh^2 \rho_1 \sqrt{h}} \Pi(\text{am}(K - \sqrt{h}\sigma), u_0, t), \\ \mu_1 &= \frac{C_1 \omega_{10}^2}{\sin^2 \theta_0 (\lambda_3 - \omega_1^2)} \int_{\sigma_0}^{\sigma} \frac{d\sigma}{1 - u_1 \xi_2} = -\frac{C_1}{\sin^2 \theta_0 \sinh^2 \rho_1 \sqrt{h}} \Pi(\text{am}(K - \sqrt{h}\sigma), u_1, t), \\ \mu_2 &= \frac{C_2 \omega_{10}^2}{\cos^2 \theta_0 (\lambda_3 - \omega_1^2)} \int_{\sigma_0}^{\sigma} \frac{d\sigma}{1 - u_1 \xi_2} = -\frac{C_2}{\cos^2 \theta_0 \sinh^2 \rho_1 \sqrt{h}} \Pi(\text{am}(K - \sqrt{h}\sigma), u_1, t), \end{aligned} \quad (62)$$

where the integration constants are chosen such that $\mu_a(\sigma_0) = 0$. The elimination of λ_2 from (50) and (59) leads to

$$C_0^2 = \frac{1}{4}\omega_1^2 \sinh^2 2\rho_0. \quad (63)$$

We choose minus sign as $C_0 = -\omega_1 \sinh 2\rho_0/2$, which yields

$$C_1 = \frac{1}{2}\omega_0 \sin^2 \theta_0 \sinh 2\rho_0, \quad C_2 = \frac{1}{2}\omega_0 \cos^2 \theta_0 \sinh 2\rho_0 \quad (64)$$

owing to (43) and (39). The expression (63) together with (48) reads

$$\frac{V + \omega_1^4}{\omega_1^4} = \frac{\sinh^2 \rho_0 \cosh^2 \rho_0 - \sinh^2 \rho_1 \cosh^2 \rho_1}{\cosh^4 \rho_1}, \quad (65)$$

which indeed satisfies (51). We use $\omega_1/\omega_0 = \coth \rho_1$ in (59) to express h as

$$h = \frac{\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1}{\sinh^2 \rho_1} \omega_0^2 \quad (66)$$

and obtain

$$\begin{aligned} \mu_0 &= -\frac{\sinh 2\rho_0}{\sqrt{2} \cosh \rho_1 \sqrt{\cosh 2\rho_1 + \cosh 2\rho_0}} \Pi(\text{am}(K - \sqrt{h}\sigma), u_0, t), \\ \mu_1 &= \mu_2 = -\frac{\sinh 2\rho_0}{\sqrt{2} \sinh \rho_1 \sqrt{\cosh 2\rho_1 + \cosh 2\rho_0}} \Pi(\text{am}(K - \sqrt{h}\sigma), u_1, t), \end{aligned} \quad (67)$$

where u_0 and u_1 defined in (56) are described by

$$u_0 = \frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 + 1}, \quad u_1 = \frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 - 1} \quad (68)$$

and the equality $\mu_1 = \mu_2$ is due to (64). The condition (60) together with (66) implies

$$\omega_1 = \frac{\cosh \rho_1}{\sqrt{\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1}} \frac{n_1 K(t)}{\pi}, \quad (69)$$

which is compared to (31).

To demand that the string is closed at constant global time t we substitute $\tau = (t - \mu_0(\sigma))/\omega_0$ into the ansatz for ϕ_1, ϕ_2 in (41) and (34) to obtain

$$\phi_i(t, \sigma) = \frac{\omega_i}{\omega_0} t + \mu_i(\sigma) - \frac{\omega_i}{\omega_0} \mu_0(\sigma), \quad i = 1, 2. \quad (70)$$

Introducing two integer winding numbers m_i for the angular coordinates ϕ_i we have the following closeness conditions

$$2\pi m_i = \delta\phi_i = \delta\mu_i - \frac{\omega_i}{\omega_0} \delta\mu_0, \quad (71)$$

where $\delta\phi_i = \phi_i(t, 2\pi) - \phi_i(t, 0)$, $\delta\mu_a = \mu_a(2\pi) - \mu_a(0)$. Owing to $\omega_1 = \omega_2, \mu_1 = \mu_2$ they are expressed in terms of the same winding number $m_1 = m_2 = m$ in a single form as

$$\begin{aligned} 2\pi m &= 2n_1 \Delta\phi, \\ \Delta\phi &= \frac{\sinh 2\rho_0}{\sqrt{2} \sinh \rho_1 \sqrt{\cosh 2\rho_1 + \cosh 2\rho_0}} (\Pi(u_1, t) - \Pi(u_0, t)). \end{aligned} \quad (72)$$

Thus m is described by two parameters ρ_0, ρ_1 for fixed n_1 and the parameter θ_0 does not appear.

From (56) and (55) the energy \mathcal{E} and total spin $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ can be computed to be

$$\begin{aligned} \mathcal{E} &= \frac{\omega_0(\lambda_3 - \omega_0^2)}{\omega_{10}^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} (1 - u_0 \xi_2) = \frac{n_1}{\pi} \frac{\omega_0}{\omega_{10}^2 \sqrt{h}} (\lambda_{31} E - (\omega_0^2 - \lambda_1) K), \\ \mathcal{S} &= \frac{\omega_1(\lambda_3 - \omega_1^2)}{\omega_{10}^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} (1 - u_1 \xi_2) = \frac{n_1}{\pi} \frac{\omega_1}{\omega_{10}^2 \sqrt{h}} (\lambda_{31} E - (\omega_1^2 - \lambda_1) K), \end{aligned} \quad (73)$$

where two spins are not equal as $\mathcal{S}_1 = \sin^2 \theta_0 \mathcal{S}, \mathcal{S}_2 = \cos^2 \theta_0 \mathcal{S}$, but have a fixed ratio $\mathcal{S}_1/\mathcal{S}_2 = \tan^2 \theta_0$. They are further written by

$$\begin{aligned} \mathcal{E} &= \frac{n_1}{\pi} \frac{\sinh \rho_1}{\sqrt{\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1}} ((\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1) E - \sinh^2 \rho_0 K), \\ \mathcal{S} &= \frac{n_1}{\pi} \frac{\cosh \rho_1}{\sqrt{\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1}} ((\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1) E - \cosh^2 \rho_0 K). \end{aligned} \quad (74)$$

Thus the large spin limit is specified by $t \approx 1$, that is, the long string limit $\rho_1 \rightarrow \infty$ so that $u_0 \approx u_1 \approx 1$ and $\omega_0 \approx \omega_1 \gg 1$ in (69). Two suitable combinations of \mathcal{E} and \mathcal{S} lead to

$$\mathcal{E} - \frac{\omega_1}{\omega_0} \mathcal{S} = \frac{n_1}{\pi} \frac{\sqrt{\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1}}{\sinh \rho_1} (K(t) - E(t)), \quad (75)$$

$$\mathcal{E} - \frac{\omega_0}{\omega_1} \mathcal{S} = \frac{n_1}{\pi} \frac{\omega_0}{\sqrt{h}} K(t) = \frac{n_1}{\pi} \frac{\sinh \rho_1}{\sqrt{\cosh^2 \rho_1 + \cosh^2 \rho_0 - 1}} K(t). \quad (76)$$

We parametrize the large spin limit as $\eta \rightarrow 0$ for $\omega_1/\omega_0 = 1 + \eta$ ($\eta \geq 0$) and use $K(t) \approx 1/2 \ln(16/(1-t)) - (1-t)/8 \ln(1-t) + \dots$ with $t \approx 1 - 2 \cosh 2\rho_0 \eta$ and $\mathcal{S} \approx n_1/(2\pi\eta)$ for (76) to have

$$\mathcal{E} - \mathcal{S} + \frac{n_1}{2\pi} \approx \frac{n_1}{2\pi} \left(\ln \frac{16\pi\mathcal{S}}{n_1} - \ln \cosh 2\rho_0 \right), \quad (77)$$

which includes the subleading contributions. In the large spin limit $u_0 \approx u_1 \approx t \approx 1$ in $t \leq u_0, t \leq u_1$ which are associated with the range (53), the equal angular separation $\Delta\phi$ (72) for one segment turns out to be

$$\Delta\phi = \frac{m}{n_1} \pi \approx \tan^{-1} \left(\frac{1}{\sinh 2\rho_0} \right), \quad (78)$$

where the following formula is used

$$\Pi(u, t) \approx \sqrt{\frac{u}{(1-u)(u-t)}} \left(\frac{\pi}{2} - \sin^{-1} \sqrt{\frac{1-u}{1-t}} \right) \quad (79)$$

for $u \approx t \approx 1, t \leq u$. Substitution of (78) into (77) yields

$$E - S \approx \frac{n_1 \sqrt{\lambda}}{2\pi} \left(\ln \frac{16\pi S}{n_1 \sqrt{\lambda}} - 1 + \ln \sin \frac{m\pi}{n_1} \right). \quad (80)$$

This expression for the spiky string solution with two spins is compared with (33) for the string solution of round arc shape with two equal spins. The coefficients of the subleading $\ln \sin(n_2\pi/n_1)$ and $\ln \sin(m\pi/n_1)$ terms are different where n_2 is the winding number in the θ direction while m is the identical winding number in both ϕ_1 and ϕ_2 directions. The energy-spin relation (80) for the AdS_5 spiky string with total spin S shows the same expression as for the AdS_3 spiky string with one spin S_1 and the winding number m_1 in the ϕ_1 direction [24, 21, 23]

$$E - S_1 \approx \frac{n_1 \sqrt{\lambda}}{2\pi} \left(\ln \frac{16\pi S_1}{n_1 \sqrt{\lambda}} - 1 + \ln \sin \frac{m_1\pi}{n_1} \right). \quad (81)$$

4 Conclusion

By means of reduction procedure of the string sigma model to the 1-d Neumann integrable model [38] we have demonstrated that the string solution [34, 35, 36] with two equal spins in AdS_5 with three angular coordinates (θ, ϕ_1, ϕ_2) is reproduced to be of round arc shape in the (ρ, θ) plane, where the string stretches in both ρ and θ directions and is characterized by the arc number and the winding number in the θ direction. In this demonstration we have seen that the two constants of motion b_1, b_2 and the three frequencies $\omega_0, \omega_1 = \omega_2$ are expressed in terms of two independent parameters, the minimum and maximum AdS radial coordinates ρ_0, ρ_1 .

Based on the more general reduction procedure to the integrable Neumann-Rosochatius system [39, 40] we have constructed the spiky string solution with two spins in AdS_5 by being guided from the analysis of string equations of motion for the global coordinates in the string sigma-model action itself. In spite of starting with the equal choice of two frequencies $\omega_1 = \omega_2$ associated with the two angular directions ϕ_1, ϕ_2 we have a spiky string solution with two unequal spins whose ratio is fixed. We have observed that the two winding numbers in the ϕ_1, ϕ_2 directions should be the same. We have demonstrated that the constant of motion V , the three conserved momenta C_a and the three frequencies ω_a are also expressed in terms of two independent parameters ρ_0, ρ_1 . The energy and total spin as well as two identical winding numbers are described by two independent parameters ρ_0, ρ_1 and the arc number n_1 . The string configuration consisting of n_1 arcs with a fixed angular separation $2\Delta\phi$ for one arc in the (ρ, ϕ_1) plane is the same as that in the (ρ, ϕ_2) plane, where the spiky string stretches in ρ as well as in both ϕ_1 and ϕ_2 directions.

We have observed that the energy-spin relation in the large spin limit for the spiky string solution with two spins and two identical winding numbers in AdS_5 shows the energy growing

logarithmically with the total spin and becomes the same form as for the spiky string with one spin and one winding number in AdS_3 , while it shows a slight difference in the coefficient of a subleading term compared with the energy-spin relation for the string of round arc shape with two equal spins in AdS_5 .

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